

# HAUSDORFF DIMENSION AND NON-DEGENERATE FAMILIES OF PROJECTIONS

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ABSTRACT. We study parametrized families of orthogonal projections for which the dimension of the parameter space is strictly less than that of the Grassmann manifold. We answer the natural question of how much the Hausdorff dimension may decrease by verifying the best possible lower bound for the dimension of almost all projections of a finite measure. We also show that a similar result is valid for smooth families of maps from  $n$ -dimensional Euclidean space to  $m$ -dimensional one.

## 1. INTRODUCTION

The behaviour of different concepts of dimensions of sets and measures under projections has been investigated intensively for several decades. The study was initiated by Marstrand [Mar] in the 1950's. Mattila [Mat1] considered Hausdorff dimension of sets in the 1970's, and in the late 1980's and in the 1990's several authors contributed to the field. In 2000 Peres and Schlag [PS] proved a very general result concerning transversal families of mappings and Sobolev dimension. For a more detailed account of the history, see the survey of Mattila [Mat3].

All the above results concerning Hausdorff dimension may be simplified by stating that the dimension is preserved under almost all projections. The essential assumption is transversality which is guaranteed in many cases by identifying the parameter space with an open subset of the Grassmann manifold. The question we are addressing is that how much the dimension may drop under almost all projections provided that the dimension of the parameter space is less than that of the Grassmann manifold. The following conclusion can be drawn from [PS]: Fubini's theorem implies that for a given set or a measure the dimension is preserved for almost all projections in almost all  $k$ -dimensional families for any  $k$ . Hence, for a given measure one obtains information for typical families. However, in general there is no way to conclude whether a given family is typical for a given measure. Furthermore, the results of [PS] concerning exceptional sets

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of parameters may be applied if  $k$  is large enough but the bounds obtained in this way are not optimal except in a few special cases (see Remark 4.4).

The aforementioned question appears naturally in applications. For example, the study of projections of measures invariant under the geodesic flow on  $n$ -dimensional Riemann manifolds leads to a 1-dimensional family of projections from a  $2(n-1)$ -dimensional space onto an  $(n-1)$ -dimensional space (see [LL, JLL]). Another interesting example is Falconer's [Fa2] attempt to prove that there are no Besicovitch  $(n, m)$ -sets for  $m \geq 2$ . A set  $A \subset \mathbb{R}^n$  is a Besicovitch  $(n, m)$ -set if the  $n$ -dimensional Lebesgue measure of  $A$  is zero and  $A$  contains a translate of every  $m$ -dimensional linear subspace of  $\mathbb{R}^n$ . There is a gap in the proof related to the issue of the behaviour of the dimension under a  $k$ -dimensional family of projections onto  $m$ -planes where  $k$  is less than the dimension of the Grassmann manifold  $G(n, m)$ .

To obtain results for almost all projections it is not sufficient to assume that the projection family is smooth since it is possible to parametrize exceptional projections with many parameters. To prevent this from happening, we assume that the family is locally embeddable into the Grassmann manifold guaranteeing that the mapping is changed when the parameter is changed.

The cases of 1-dimensional families of projections onto  $m$ -planes and general families of projections onto lines or hyper-planes are dealt in [JJLL]. In this paper we solve completely the general case by proving the best possible almost sure lower bound in a  $k$ -dimensional family of projections onto  $m$ -planes in  $\mathbb{R}^n$  (see Theorem 3.2). We also verify that the corresponding result is valid for parametrized families of smooth maps between  $\mathbb{R}^n$  and  $\mathbb{R}^m$  (see Theorem 4.3).

When applying our result to the setting of [Fa2] we observe that the dimension of the parameter space is too small to obtain the desired result except in the case of Besicovitch  $(n, n-1)$ -sets. Since our result is the best possible one for general families, this means that if there is a way to fix the gap in [Fa2] for Besicovitch  $(n, m)$ -sets with  $m < n-1$ , one needs to utilize the special properties of the projection family constructed in [Fa2].

The paper is organized as follows. In Section 2 we give the basic definitions and the auxiliary results needed later. Our main theorem concerning families of projections is verified in Section 3 and generalized to families of smooth maps in Section 4.

## 2. BASIC DEFINITIONS

In this section we introduce the notation used throughout this paper. Let  $m$  and  $n$  be integers with  $0 < m < n$  and let  $\mu$  be a finite Radon measure on  $\mathbb{R}^n$  with compact support. The Hausdorff dimension  $\dim$  of  $\mu$  is defined in terms of local dimensions as follows:

$$(2.1) \quad \dim \mu = \sup \{s \geq 0 \mid \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \geq s \text{ for } \mu\text{-almost all } x \in \mathbb{R}^n\},$$

where  $B(x, r)$  is the open ball with centre at  $x$  and radius  $r$ . Equivalently,

$$(2.2) \quad \dim \mu = \inf\{\dim A \mid A \subset \mathbb{R}^n \text{ is a Borel set with } \mu(A) > 0\}.$$

For this equivalence and other properties of dimensions of measures see [Fa3, Proposition 10.2]. It follows easily from (2.1) that

$$(2.3) \quad I_t(\mu) < \infty \implies \dim \mu \geq t,$$

where

$$I_t(\mu) = \iint |x - y|^{-t} d\mu(x) d\mu(y)$$

is the  $t$ -energy of  $\mu$ .

Let  $k$  be an integer with  $0 < k < m(n - m)$ . Note that  $m(n - m)$  is the dimension of the Grassmann manifold  $G(n, m)$  of all  $m$ -dimensional linear subspaces of  $\mathbb{R}^n$ . Supposing that  $\Lambda \subset \mathbb{R}^k$  is open, we consider parametrized families  $\{F_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m \mid \lambda \in \Lambda\}$  of smooth maps. We denote the orthogonal projection in  $\mathbb{R}^n$  onto an  $m$ -dimensional subspace  $V \in G(n, m)$  by  $\Pi_V$ . When investigating parametrized families of orthogonal projections  $\Pi_{V_\lambda} : \mathbb{R}^n \rightarrow V_\lambda$  onto  $V_\lambda \in G(n, m)$ , we consider them as mappings from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . Clearly, such a family could also be viewed as a family from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  by identifying the range  $V_\lambda \in G(n, m)$  with  $\mathbb{R}^m$  in a systematic manner, for example, by fixing an orthonormal basis of some  $V_{\lambda_0}$  and by rotating the basis to  $V_\lambda$  by a rotation which rotates  $V_\lambda$  to  $V_{\lambda_0}$ . Since the identification is neither unique nor essential, we omit it.

The image of a measure  $\mu$  under a map  $T : X \rightarrow Y$  is denoted by  $T_*\mu$ , that is,  $T_*\mu(A) = \mu(T^{-1}(A))$  for all  $A \subset Y$ . If  $\mu$  is a Radon measure on  $X$  with compact support and  $T$  is a Lipschitz map, the image measure  $T_*\mu$  is a Radon measure on  $Y$  with compact support [Mat2, Theorem 1.18]. We use the notation  $\text{spt } \mu$  for the support of a measure  $\mu$ . Obviously,

$$(2.4) \quad \dim \mu - (n - m) \leq \dim(\Pi_V)_*\mu \leq \min\{\dim \mu, m\}$$

for all  $V \in G(n, m)$ .

For  $r = 2, \dots, n$  a simple  $r$ -vector in  $\mathbb{R}^n$  is denoted by  $v_1 \wedge \dots \wedge v_r$  where  $v_i \in \mathbb{R}^n$  for  $i = 1, \dots, r$ . A non-zero simple  $r$ -vector  $v_1 \wedge \dots \wedge v_r$  determines uniquely an  $r$ -plane  $\langle v_1, \dots, v_r \rangle \in G(n, r)$  (see [Fe, Section 1.6]). The norm of a simple  $r$ -vector is given by

$$\|v_1 \wedge \dots \wedge v_r\| = \sqrt{\det(DD^T)}$$

where  $D$  is the  $r \times n$ -matrix whose  $i^{\text{th}}$  row consists of the coordinates of  $v_i$ . Note that the norm is equal to the  $r$ -dimensional volume of the parallelepiped spanned by  $v_1, \dots, v_r$ . In particular, if the vectors  $v_1, \dots, v_l \in \mathbb{R}^n$  are perpendicular to the vectors  $u_1, \dots, u_t \in \mathbb{R}^n$ , we have

$$(2.5) \quad \|v_1 \wedge \dots \wedge v_l \wedge u_1 \wedge \dots \wedge u_t\| = \|v_1 \wedge \dots \wedge v_l\| \cdot \|u_1 \wedge \dots \wedge u_t\|.$$

A linear map  $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  can be naturally extended to a linear map  $\wedge_r L : \wedge_r \mathbb{R}^n \rightarrow \wedge_r \mathbb{R}^m$  between the vector spaces of  $r$ -vectors. The norm of  $\wedge_r L$  is defined by

$$\|\wedge_r L\| = \sup\{\|\wedge_r L(\xi)\| \mid \xi \text{ is a simple } r\text{-vector with } \|\xi\| = 1\}.$$

Note that  $\|\wedge_n L\| = |\det L|$ .

The following well-known lemma plays a fundamental role in our approach. We use the notation  $\mathcal{L}^k$  for the Lebesgue measure on  $\mathbb{R}^k$ . In the case  $k = 1$  the Lebesgue measure is denoted by  $\mathcal{L}$ .

**Lemma 2.1.** *Let  $n, m, k$  and  $l$  be integers satisfying  $0 < k < m(n - m)$  and  $l \geq m$ . Let  $\Lambda \subset \mathbb{R}^k$  be bounded and let  $\{F_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^l \mid \lambda \in \Lambda\}$  be a parametrized family of Lipschitz maps such that for all  $\lambda \in \Lambda$  there exists a smooth  $m$ -dimensional submanifold of  $\mathbb{R}^l$  containing  $F_\lambda(\mathbb{R}^n)$ . Assume that  $\mu$  is a finite Radon measure on  $\mathbb{R}^n$  with compact support and  $r$  is a positive real number such that  $r \leq m$ . Suppose that for all  $z \in \text{spt } \mu$  there exist  $\varepsilon > 0$  and  $C > 0$  such that for all  $x \neq y \in B(z, \varepsilon)$  and for all  $\delta > 0$*

$$(2.6) \quad \mathcal{L}^k(\{\lambda \in \Lambda \mid |F_\lambda(x) - F_\lambda(y)| \leq \delta\}) \leq C\delta^r |x - y|^{-r}.$$

*Then  $\dim(F_\lambda)_* \mu = \dim \mu$  for  $\mathcal{L}^k$ -almost all  $\lambda \in \Lambda$  provided that  $\dim \mu \leq r$ . Furthermore,  $\dim(F_\lambda)_* \mu \geq r$  for  $\mathcal{L}^k$ -almost all  $\lambda \in \Lambda$  provided that  $\dim \mu > r$ . Finally, for  $\mathcal{L}^k$ -almost all  $\lambda \in \Lambda$  the image measure  $(F_\lambda)_* \mu$  is absolutely continuous with respect to the  $m$ -dimensional Hausdorff measure  $\mathcal{H}^m$  if  $\dim \mu > m$  and  $r = m$ .*

*Proof.* Covering the compact set  $\text{spt } \mu$  by a finite collection of open balls  $B(z_i, \varepsilon_i)$  and letting  $\mu_i = \mu|_{B(z_i, \varepsilon_i)}$  be the restriction of  $\mu$  to the ball  $B(z_i, \varepsilon_i)$ , we have  $\dim \mu = \min_i \dim \mu_i$  and  $\dim(F_\lambda)_* \mu = \min_i \dim(F_\lambda)_* \mu_i$ . Therefore, we may restrict our consideration to a restricted measure  $\mu_i$ .

The first two claims follow similarly as in [JJLL, Lemmas 2.1 and 2.2]. Even though [JJLL] deals with projections the only essential assumption is that  $(F_\lambda)_* \mu$  is a Radon measure.

For the last claim proceed as in the proof of [JJLL, Lemma 2.2] to find a restriction of  $\mu$  having finite  $m$ -energy and apply the proof of [Mat2, Theorem 9.7]. Here we use the assumption that the range of  $F_\lambda$  is contained in a smooth  $m$ -dimensional submanifold  $M_\lambda$  which implies that  $\mathcal{H}^m|_{M_\lambda}(B(x, r))$  is comparable to  $r^m$ .  $\square$

*Remark 2.2.* In the proof of Theorem 3.2 we need local coordinates on  $G(n, m)$  and the following choice turns out to be useful. Consider  $V \in G(n, m)$ . Let  $\{e_1, \dots, e_m\}$  and  $\{e_{m+1}, \dots, e_n\}$  be orthonormal bases of  $V$  and its orthogonal complement  $V^\perp \in G(n, n - m)$ , respectively. One may choose local coordinates on  $G(n, m)$  near  $V$  in terms of rotations of the basis vectors  $\{e_1, \dots, e_m\}$  in the following manner: For  $i = 1, \dots, m$  and  $j = m + 1, \dots, n$ , let  $-\frac{\pi}{4} < \alpha_{ij} < \frac{\pi}{4}$

be the components of  $\alpha \in ]-\frac{\pi}{4}, \frac{\pi}{4}[^{m(n-m)}$ . Rotating  $e_i$  by the angle  $\alpha_{ij}$  towards  $e_j$  for all  $i$  and  $j$  gives local coordinates for the  $m$ -plane  $V(\alpha)$  spanned by the rotated vectors. More precisely,  $V(\alpha) = \langle e_1(\alpha), \dots, e_m(\alpha) \rangle$ , where  $e_i(\alpha) = \prod_{j=m+1}^n R^{ij}(\alpha_{ij})e_i$  is an ordered product for all  $i = 1, \dots, m$  and

$$(R^{ij}(\beta)x)_l = \begin{cases} x_i \cos \beta - x_j \sin \beta, & \text{if } l = i \\ x_i \sin \beta + x_j \cos \beta, & \text{if } l = j \\ x_l, & \text{otherwise.} \end{cases}$$

For the proof of the fact that these rotations give local coordinates, see [JJLL, Remark 2.4]. Further, let  $\{\frac{\partial}{\partial \alpha_{ij}} \mid i = 1, \dots, m, j = m+1, \dots, n\}$  be the basis of the tangent space  $T_V G(n, m)$  obtained in this way. A straightforward calculation shows that for any  $z \in V^\perp$ ,  $w \in V$ ,  $i \in \{1, \dots, m\}$  and  $j \in \{m+1, \dots, n\}$  we have

$$(2.7) \quad \left. \frac{\partial \Pi_{V(\alpha)}(z)}{\partial \alpha_{ij}} \right|_{\alpha=0} = z_j e_i \quad \text{and} \quad \left. \frac{\partial \Pi_{V(\alpha)}(w)}{\partial \alpha_{ij}} \right|_{\alpha=0} = w_i e_j.$$

### 3. FAMILIES OF PROJECTIONS

In this section we state and prove our main theorem concerning parametrized families of orthogonal projections. We equip the Grassmann manifold  $G(n, m)$  with a Riemann metric and define the class of families of projections we are working with.

**Definition 3.1.** Let  $\Lambda \subset \mathbb{R}^k$  be open. A parametrized family  $\{\Pi_{V_\lambda} \mid \lambda \in \Lambda\}$  of orthogonal projections in  $\mathbb{R}^n$  onto  $m$ -planes is called *non-degenerate* if the mapping  $\lambda \mapsto V_\lambda$  is continuously differentiable and the derivative  $D_\lambda V_\lambda$  is injective for all  $\lambda \in \Lambda$ .

For all  $x \in \mathbb{R}$ , we denote by  $\lfloor x \rfloor$  the smallest integer  $q \geq 0$  such that  $x \leq q$ . Furthermore, given an integer  $0 < k < m(n-m)$ , define

$$(3.1) \quad p(l) = n - m - \left\lfloor \frac{k - l(n-m)}{m-l} \right\rfloor$$

for  $l = 0, \dots, m-1$ .

**Theorem 3.2.** Let  $\Lambda \subset \mathbb{R}^k$  be an open set and let  $\mu$  be a finite Radon measure on  $\mathbb{R}^n$  with compact support. Assume that a family  $\{\Pi_{V_\lambda} \mid \lambda \in \Lambda\}$  of orthogonal projections in  $\mathbb{R}^n$  onto  $m$ -planes is non-degenerate. Then for all  $l = 0, \dots, m-1$  and for  $\mathcal{L}^k$ -almost all  $\lambda \in \Lambda$

$$(3.2) \quad \dim(\Pi_{V_\lambda})_* \mu \geq \begin{cases} \dim \mu - p(l), & \text{if } p(l) + l \leq \dim \mu \leq p(l) + l + 1, \\ l + 1, & \text{if } p(l) + l + 1 \leq \dim \mu \leq p(l + 1) + l + 1. \end{cases}$$

Furthermore, for  $\mathcal{L}^k$ -almost all  $\lambda \in \Lambda$  the projected measure  $(\Pi_{V_\lambda})_*\mu$  is absolutely continuous with respect to  $\mathcal{H}^m$  provided that  $\dim \mu > p(m-1) + m$ . The lower bounds given in (3.2) and the condition for the absolute continuity are the best possible ones.

*Remark 3.3.* a) Theorem 3.2 is valid if  $D_\lambda V_\lambda$  is injective only for  $\mathcal{L}^k$ -almost all  $\lambda \in \Lambda$  since, by continuity, the set  $N = \{\lambda \in \Lambda \mid D_\lambda V_\lambda \text{ is not injective}\}$  is closed, and therefore, one may replace  $\Lambda$  by  $\Lambda \setminus N$ .

b) The injectivity assumption is natural: Theorem 3.2 is not necessarily true without it. Indeed, by the sharpness of (3.2), there is a  $(k-1)$ -dimensional family for which the lower bound in (3.2) is obtained. We extend the family to a  $k$ -dimensional one by adding an extra parameter which does not change the maps. The extension does not affect the dimensions of the projections, and it follows from (3.1) that (3.2) is not valid for the extended family for which the injectivity fails.

c) The fact that the function  $p$  in (3.1) is increasing can be gleaned from Figure 1. Indeed, after filling the  $l$  lowest rows in Figure 1 with dots, one is left with  $k - l(n-m)$  dots, where  $k$  is the original number of dots. Proceed by filling the columns from left. The number of the columns needed is  $\lceil \frac{k-l(n-m)}{m-l} \rceil$  implying that  $p(l)$  is the number of the remaining unoccupied columns. When increasing  $l$  by one, one needs to move dots from the last occupied column to the unoccupied slots on the  $(l+1)^{\text{th}}$  row. This means that the number of unoccupied columns may increase but not decrease.

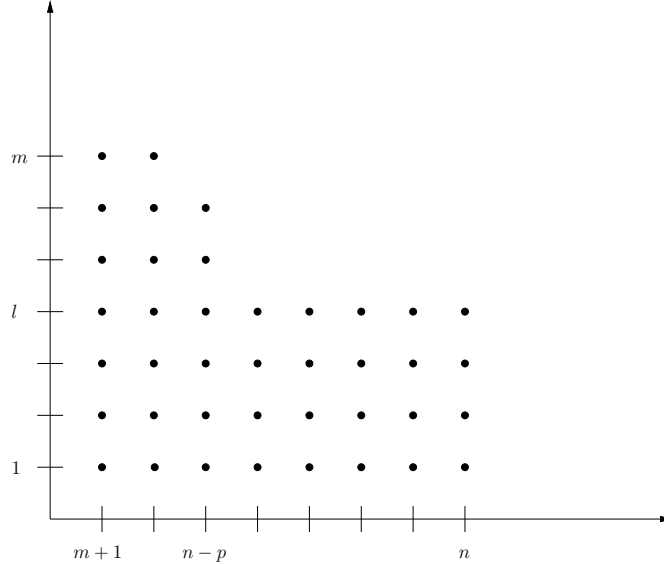


FIGURE 1. The dotting that explains the definition of  $p(l)$  and the sharpness of the lower bounds in Theorem 3.2.

We continue by proving a technical lemma.

**Lemma 3.4.** *Let  $A_1, \dots, A_k : \mathbb{R}^{n-m} \rightarrow \mathbb{R}^m$  be linear maps and let  $C, d > 0$ . Assume that  $\|A_i\| < C$  for all  $i = 1, \dots, k$  and  $\|A_1 \wedge \dots \wedge A_k\| > d$  where  $A_1, \dots, A_k$  are considered as vectors in  $\mathbb{R}^{m(n-m)}$ . Suppose that for some integers  $1 \leq t \leq n - m$  and  $0 \leq l \leq m - 1$  we have  $k > m(t - 1) + l(n - m - t + 1)$ . Then there exist  $d'$  depending only on  $C, d$  and  $n$ , and a  $t$ -dimensional subspace  $W \subset \mathbb{R}^{n-m}$  such that for all  $z \in W \setminus \{0\}$  there are  $j_1, \dots, j_{l+1}$  satisfying*

$$(3.3) \quad \|A_{j_1}(z) \wedge \dots \wedge A_{j_{l+1}}(z)\| > d'|z|^{l+1}.$$

*In particular,  $\dim \langle A_1(z), \dots, A_k(z) \rangle \geq l + 1$  for all  $z \in W \setminus \{0\}$ .*

*Proof.* The heuristic idea behind the proof is as follows: assuming that the last claim is not true and using the fact that a linear map is uniquely determined by the images of the basis vectors, one can find  $n - m - t + 1$  orthonormal vectors having at most  $l$  linearly independent images. Since the remaining  $t - 1$  basis vectors have at most  $m$  linearly independent images, there are at most  $m(t - 1) + l(n - m - t + 1) < k$  independent maps in the family  $\{A_1, \dots, A_k\}$  which is a contradiction. Hence, the last claim holds which, in turn, implies (3.3) since the left hand side of (3.3) is continuous and homogeneous of degree  $l + 1$ .

To make the above idea rigorous, assume that for some  $\tilde{d} > 0$  and for all  $t$ -dimensional subspaces  $W \subset \mathbb{R}^{n-m}$  there is  $z \in W \setminus \{0\}$  such that the inequality  $\|A_{j_1}(z) \wedge \dots \wedge A_{j_{l+1}}(z)\| \leq \tilde{d}|z|^{l+1}$  holds for all  $j_1, \dots, j_{l+1}$ . By homogeneity, one can find inductively orthonormal vectors  $z_1, \dots, z_{n-m-t+1}$  with

$$(3.4) \quad \|A_{j_1}(z_i) \wedge \dots \wedge A_{j_{l+1}}(z_i)\| \leq \tilde{d}$$

for all  $i = 1, \dots, n - m - t + 1$  and for all  $j_1, \dots, j_{l+1}$ . We will prove that (3.4) implies  $\tilde{d} > \frac{d}{C'}$  where  $C'$  depends only on  $C$  and  $n$ . This means that (3.3) holds with  $d' = \frac{d}{C'}$  since otherwise by taking  $\tilde{d} = d'$  we would get a contradiction.

Extend  $\{z_1, \dots, z_{n-m-t+1}\}$  to an orthonormal basis  $\{z_1, \dots, z_{n-m}\}$  of  $\mathbb{R}^{n-m}$ , fix an orthonormal basis in  $\mathbb{R}^m$ , and view  $A_j$  as an  $m \times (n - m)$ -matrix determined by these bases. Let  $D$  be the  $k \times m(n - m)$ -matrix whose  $j^{\text{th}}$  row consists of the elements of  $A_j$ . According to the Cauchy-Binet formula

$$\|A_1 \wedge \dots \wedge A_k\| = \sqrt{\det(DD^T)} = \sqrt{\sum_{d_k} d_k^2}$$

where the sum is over all  $k \times k$ -minors  $d_k$  of  $D$ . Since  $k > m(t - 1) + l(n - m - t + 1)$ , the pigeonhole principle implies that any  $k \times k$ -submatrix of  $D$  contains at least  $l + 1$  columns picked out from the set of columns determined by  $A_1(z_i)$  for some  $i = 1, \dots, n - m - t + 1$ . Applying the Cauchy-Binet formula in (3.4) gives that any  $(l + 1) \times (l + 1)$ -minor picked out from these  $l + 1$  columns has absolute value at most  $\tilde{d}$ , and therefore, every term in the expression of any minor  $d_k$  contains

a factor at most  $\tilde{d}$ . From the fact that  $\|A_i\| \leq C$  for all  $i = 1, \dots, k$ , we derive

$$d < \|A_1 \wedge \dots \wedge A_k\| \leq C' \tilde{d}$$

where  $C'$  depends on  $C, k, l, n$  and  $m$ . Since  $l < m < n$  and  $k \leq m(n - m)$ , we may choose  $C'$  in such a way that it depends only on  $C$  and  $n$ . Therefore, as we claimed  $\tilde{d} > \frac{d}{C'}$ , which completes the proof of (3.3).

Finally, the last claim follows since  $\dim \langle A_{j_1}(z), \dots, A_{j_{l+1}}(z) \rangle = l + 1$ .  $\square$

*Remark 3.5.* Define  $F : G(n, m) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $F(V, z) = \Pi_V(z)$ . Given  $V_0 \in G(n, m)$ , by Remark 2.2 (see (2.7)), the formula  $\frac{\partial F(V_0, \cdot)}{\partial \alpha_{ij}}$  defines a linear map from  $V_0^\perp$  to  $V_0$ , where  $\alpha_{ij}$  are the local coordinates around  $V_0$  defined in Remark 2.2. The maps  $\frac{\partial F(V_0, \cdot)}{\partial \alpha_{ij}}$  are clearly linearly independent for  $i = 1, \dots, m$  and  $j = m + 1, \dots, n$ , and since  $\dim G(n, m) = m(n - m)$ , all linear maps from  $V_0^\perp$  to  $V_0$  are linear combinations of them. In particular, the map defined by  $D_V F(V_0, \cdot)(v)$  for all  $v \in T_{V_0} G(n, m)$  is a bijection from  $T_{V_0} G(n, m)$  onto the space of linear maps from  $V_0^\perp$  to  $V_0$ . Linear maps from  $V_0$  to  $V_0^\perp$  can be characterized similarly.

Let  $\lambda^0 \in \Lambda \subset \mathbb{R}^k$ ,  $z_1 \in V_{\lambda^0}$  and  $z_2 \in V_{\lambda^0}^\perp$ . By the above observation

$$(3.5) \quad D_\lambda \Pi_{V_{\lambda^0}}(z_1)(w) \in V_{\lambda^0}^\perp \text{ and } D_\lambda \Pi_{V_{\lambda^0}}(z_2)(w) \in V_{\lambda^0}$$

for all  $w \in \mathbb{R}^k$ , where  $D_\lambda \Pi_{V_{\lambda^0}}(z)$  is the derivative of the map  $\lambda \mapsto \Pi_{V_\lambda}(z)$  at  $\lambda^0$ . Since  $D_\lambda \Pi_{V_{\lambda^0}}(z_1 + z_2)(w) = D_\lambda \Pi_{V_{\lambda^0}}(z_1)(w) + D_\lambda \Pi_{V_{\lambda^0}}(z_2)(w)$ , condition (3.5) implies that

$$\Pi_{V_{\lambda^0}}(\wedge_r D_\lambda \Pi_{V_{\lambda^0}}(z_1 + z_2)(\xi)) = \wedge_r D_\lambda \Pi_{V_{\lambda^0}}(z_2)(\xi)$$

for any  $r \leq m$  and for any simple  $r$ -vector  $\xi$  on  $\mathbb{R}^k$ . Therefore, we have for any  $r \leq m$  that

$$(3.6) \quad \|\wedge_r D_\lambda \Pi_{V_{\lambda^0}}(z_1 + z_2)\| \geq \|\wedge_r D_\lambda \Pi_{V_{\lambda^0}}(z_2)\|.$$

We will be working with families for which the transversality condition [PS, Definition 7.2] is not valid. The following proposition, which may be regarded as a partial transversality condition, is our main tool. In Proposition 3.6 the projection family does not need to be non-degenerate. Since our main interest is the case  $r < m$  and  $r < k$ , we cannot apply directly the area formula [Fe, Theorem 3.2.3] or the coarea formula [Fe, Theorem 3.2.11].

**Proposition 3.6.** *Let  $\Lambda \subset \mathbb{R}^k$  be an open set and let  $\{\Pi_{V_\lambda} \mid \lambda \in \Lambda\}$  be a family of orthogonal projections in  $\mathbb{R}^n$  onto  $m$ -planes. Suppose that the mapping  $\lambda \mapsto V_\lambda$  has a uniformly continuous derivative and there exists  $C_0 > 0$  with  $\|D_\lambda V_\lambda\| < C_0$  for all  $\lambda \in \Lambda$ . Fix  $\lambda^0 \in \Lambda$ . Assume that there are  $r \leq m$  and  $d > 0$  such that for any  $z \in V_{\lambda^0}^\perp$*

$$(3.7) \quad \|\wedge_r D_\lambda \Pi_{V_{\lambda^0}}(z)\| > d|z|^r.$$



Then there exist  $C > 0$  and  $R > 0$  such that for all  $\delta > 0$  and for all  $x \neq y \in \mathbb{R}^n$  we have

$$\mathcal{L}^k(\{\lambda \in B(\lambda^0, R) \mid |\Pi_{V_\lambda}(x - y)| \leq \delta\}) \leq C\delta^r |x - y|^{-r}.$$

*Proof.* We may restrict our consideration to the case  $|x - y| = 1$  and  $0 < \delta < \delta_0$  for some  $0 < \delta_0 < \frac{1}{2}$ . Let  $z \in \mathbb{R}^n$  be such that  $|z| = 1$  and  $|\Pi_{V_{\lambda^0}}(z)| < \delta$ . Writing  $z = z_1 + z_2$  where  $z_1 = \Pi_{V_{\lambda^0}}(z) \in V_{\lambda^0}$  and  $z_2 \in V_{\lambda^0}^\perp$ , we have  $|z_2| > \frac{1}{2}$  since  $|z_1| < \delta < \frac{1}{2}$ . By (3.6) and (3.7)

$$\|\wedge_r D_\lambda \Pi_{V_{\lambda^0}}(z)\| \geq \|\wedge_r D_\lambda \Pi_{V_{\lambda^0}}(z_2)\| > 2^{-r}d,$$

implying the existence of an  $r$ -dimensional subspace  $U \subset \mathbb{R}^k$  such that the restriction of  $D_\lambda \Pi_{V_{\lambda^0}}(z)$  to  $U$  is injective and  $|\det(D_\lambda \Pi_{V_{\lambda^0}}(z)|_U)| > 2^{-r}d$ . Since the mapping  $\lambda \mapsto V_\lambda$  has uniformly continuous derivative and the mapping  $V \mapsto \Pi_V(z)$  is smooth there exists  $R' > 0$  such that the restriction of  $D_\lambda \Pi_{V_{\lambda^1}}(z)$  to  $U$  is injective (with the same lower bound for the derivative as above) for all  $\lambda^1 \in (\lambda^0 + U^\perp) \cap B(\lambda^0, R')$ . We denote by  $T^{\lambda^1}$  the restriction of the mapping  $\lambda \mapsto \Pi_{V_\lambda}(z)$  to  $(\lambda^1 + U) \cap \Lambda$ .

By the above arguments,  $C_1 < |\det DT^{\lambda^1}| < C_2$  for some constants  $C_1 > 0$  and  $C_2 > 0$ . Combining this with  $\|D_\lambda V_\lambda\| < C_0$ , we obtain that the singular values of  $DT^{\lambda^1}$  are uniformly bounded from above and below. Therefore, a quantitative version of the inverse function theorem [JJLL, Lemma 3.1] gives that there exist  $a > 0$  and  $R > 0$  such that for all  $\lambda^1 \in (\lambda^0 + U^\perp) \cap B(\lambda^0, \frac{R'}{2})$  and for all  $\lambda \in (\lambda^1 + U) \cap B(\lambda^1, \frac{R'}{2})$  the mapping  $T^{\lambda^1}$  is a diffeomorphism onto its image in  $B(\lambda, R)$ , and moreover, the inclusion

$$(3.8) \quad B(T^{\lambda^1}(\lambda), a\rho) \cap T^{\lambda^1}(B(\lambda, 3R)) \subset T^{\lambda^1}(B(\lambda, \rho))$$

is valid for all  $0 < \rho < 3R$ . For each  $\lambda^1 \in (\lambda^0 + U^\perp) \cap B(\lambda^0, R)$ , let  $\hat{\lambda}$  be a minimum point of  $|T^{\lambda^1}|$  in  $B(\lambda^1, R)$ . We may assume that  $|T^{\lambda^1}(\hat{\lambda})| \leq \delta$ . By (3.8) we have  $|T^{\lambda^1}(\lambda)| > \delta$  for any  $\lambda \in ((\lambda^1 + U) \cap B(\lambda^1, R)) \setminus B(\hat{\lambda}, 2\delta a^{-1})$ . Thus for any  $\lambda^1 \in (\lambda^0 + U^\perp) \cap B(\lambda^0, R)$  we obtain

$$\mathcal{L}^r(\{\lambda \in (\lambda^1 + U) \cap B(\lambda^0, R) \mid |\Pi_{V_\lambda}(z)| \leq \delta\}) \leq \mathcal{L}^r(B(0, 1)) \left(\frac{2}{a}\right)^r \delta^r.$$

The claim follows by Fubini's theorem.  $\square$

In the following lemma we compare projections onto  $m$ -planes to those onto certain extended  $(m + p)$ -planes.

**Lemma 3.7.** *Let  $a, b \in \mathbb{R}$  and let  $V : (a, b) \rightarrow G(n, m)$  be continuously differentiable. Assume that  $c \in (a, b)$ ,  $p$  is an integer with  $0 < p < n - m$  and  $U \subset V_c^\perp$  is a  $p$ -plane. Then there exist  $a', b' \in (a, b)$  such that  $c \in (a', b')$  and the function*

$\tilde{V} : (a', b') \rightarrow G(n, m + p)$  defined by  $\tilde{V}_s = \langle V_s, U \rangle$  for  $s \in (a', b')$  is well-defined, continuously differentiable and

$$\left. \frac{\partial \Pi_{V_s}(z)}{\partial s} \right|_{s=c} = \left. \frac{\partial \Pi_{\tilde{V}_s}(z)}{\partial s} \right|_{s=c}$$

for all  $z \in \langle V_c, U \rangle^\perp$ .

*Proof.* By the continuity of  $V$ , there exists a neighbourhood  $(a', b') \ni c$  such that  $V_s \cap U = \{0\}$  for all  $s \in (a', b')$  implying that  $\tilde{V}_s \in G(n, m + p)$  is well-defined. Clearly,  $\tilde{V}$  is continuously differentiable. Note that for  $z \in \langle V_c, U \rangle^\perp$  we have  $\Pi_{\tilde{V}_s}(z) \in U^\perp \cap \tilde{V}_s$ . Furthermore,  $U^\perp \cap \tilde{V}_s$  is an  $m$ -dimensional plane having distance of order  $\mathcal{O}(s - c)$  to  $V_s$  since the distances of  $V_c$  to  $V_s$  and to  $U^\perp \cap \tilde{V}_s$  are of order  $\mathcal{O}(s - c)$ . Combining this with the fact  $\Pi_{V_s}(z) = \Pi_{V_s}(\Pi_{\tilde{V}_s}(z))$ , we conclude that the angle  $\theta_s$  between  $\Pi_{V_s}(z)$  and  $\Pi_{\tilde{V}_s}(z)$  is of order  $\mathcal{O}(s - c)$ . Clearly, both  $|\Pi_{V_s}(z)|$  and  $|\Pi_{\tilde{V}_s}(z)|$  are of order  $\mathcal{O}(s - c)$ , and therefore,  $|\Pi_{V_s}(z) - \Pi_{\tilde{V}_s}(z)| = |\sin \theta_s \Pi_{\tilde{V}_s}(z)|$  is of order  $\mathcal{O}((s - c)^2)$  giving the claim.  $\square$

Now we are ready to prove Theorem 3.2. We will apply Lemma 2.1 to a parametrized family of projections onto  $(m + p)$ -planes for a suitable  $p$ . The role of Proposition 3.6 is to imply that the assumptions of Lemma 2.1 are valid. However, the dimension  $k$  of the parameter space is too small to guarantee the validity of the assumptions of Proposition 3.6. To overcome this problem we extend the parameter space, and for this purpose, we need the local coordinates defined in Remark 2.2.

*Proof of Theorem 3.2.* Let  $l$  be an integer with  $0 \leq l \leq m - 1$ . By (2.4) we may assume that  $p(l) < n - m$ . Writing  $p = p(l)$ , it follows from (3.1) (see also Remark 3.3 c) and Figure 1) that

$$(3.9) \quad l(n - m) + (n - m - p - 1)(m - l) < k \leq l(n - m) + (n - m - p)(m - l).$$

Consider  $\lambda^0 \in \Lambda$ . It is clearly enough to prove the claim in  $B(\lambda^0, R)$  for some  $R > 0$  such that  $B(\lambda^0, R) \subset \Lambda$ . Let  $R' > 0$  be such that  $\overline{B}(\lambda^0, R') \subset \Lambda$ , where  $\overline{B}(\lambda^0, R')$  is the closure of  $B(\lambda^0, R')$ . By Remark 3.5 the formula

$$A_i(z) = \left. \frac{\partial \Pi_{V_\lambda}(z)}{\partial \lambda_i} \right|_{\lambda=\lambda^0} = D_V F(V_{\lambda^0}, z) \circ D_\lambda V_\lambda(u_i)$$

defines a linear map  $A_i : V_{\lambda^0}^\perp \rightarrow V_{\lambda^0}$  for all  $i = 1, \dots, k$ , where  $\{u_1, \dots, u_k\}$  is the natural basis of  $\mathbb{R}^k$ . Letting  $t = n - m - p$ , we have by (3.9) that  $k > m(t - 1) + l(n - m - t + 1)$ . Since  $D_\lambda V_\lambda$  is injective and  $D_V F(V_{\lambda^0}, \cdot)$  is bijective (see Remark 3.5), the assumptions of Lemma 3.4 are valid. We denote by  $\{\hat{e}_{m+1}, \dots, \hat{e}_{m+t}\}$  an orthonormal basis of the space  $W \subset V_{\lambda^0}^\perp$  given by Lemma 3.4 and by  $\{\hat{e}_{m+1}, \dots, \hat{e}_{m+t}, \hat{e}_{m+t+1}, \dots, \hat{e}_n\}$  the extension of it to an orthonormal basis of  $V_{\lambda^0}^\perp$ . Letting  $\delta < \frac{1}{2}$ , define an extended parameter space

$\tilde{\Lambda} = B(\lambda^0, R') \times \prod_{i=m+t+1}^n \prod_{j=m+1}^{m+t} - \delta, \delta[$  and write  $\tilde{\lambda} = (\tilde{\lambda}^1, \tilde{\lambda}^2) \in \tilde{\Lambda}$  where  $\tilde{\lambda}^1 \in B(\lambda^0, R')$  and  $\tilde{\lambda}^2 \in ] - \delta, \delta[^{pt}$ .

For each  $\tilde{\lambda} \in \tilde{\Lambda}$ , define an  $(m+p)$ -dimensional plane

$$\tilde{V}_{\tilde{\lambda}} = \langle V_{\tilde{\lambda}^1}, \hat{e}_{m+t+1}(\tilde{\lambda}^2), \dots, \hat{e}_n(\tilde{\lambda}^2) \rangle$$

where  $\hat{e}_i(\tilde{\lambda}^2) = \prod_{j=m+1}^{m+t} R^{ij}((\tilde{\lambda}^2)_{ij})\hat{e}_i$  is as in Remark 2.2 for all  $i = m+t+1, \dots, n$ . Decreasing  $R'$  if necessary guarantees that the plane  $\tilde{V}_{\tilde{\lambda}}$  is  $(m+p)$ -dimensional for every  $\tilde{\lambda} \in \tilde{\Lambda}$ . In this way we obtain a  $\tilde{k}$ -dimensional family  $\tilde{V}_{\tilde{\lambda}}$  of  $(m+p)$ -planes, where  $\tilde{k} = k + pt$ . For  $i = 1, \dots, \tilde{k}$ , the indices  $1, \dots, k$  correspond to  $\tilde{\lambda}^1$  and the remaining indices  $k+1, \dots, \tilde{k}$  correspond to  $\tilde{\lambda}^2$ .

Note that  $\tilde{V}_{\tilde{\lambda}^0}^\perp = W$  where  $\tilde{\lambda}^0 = (\lambda^0, 0)$ . Since  $\hat{e}_j(\tilde{\lambda})$  is independent of  $\tilde{\lambda}^1$  for  $j = m+t+1, \dots, n$  we conclude from Lemma 3.7 that

$$(3.10) \quad \left. \frac{\partial \Pi_{\tilde{V}_{\tilde{\lambda}}}(z)}{\partial \tilde{\lambda}_i} \right|_{\tilde{\lambda}=\tilde{\lambda}^0} = \left. \frac{\partial \Pi_{V_{\lambda}}(z)}{\partial \lambda_i} \right|_{\lambda=\lambda^0} = A_i(z) \in V_{\lambda^0} \subset \tilde{V}_{\tilde{\lambda}^0}$$

for all  $z \in W$  and for all  $i \in \{1, \dots, k\}$ . Fix  $z \in W$  and let  $j_1, \dots, j_{l+1} \in \{1, \dots, k\}$  be the indices for which (3.3) is satisfied with  $d' > 0$ . Let  $|z_{j_0}| = \max_j \{|z_j|\}$ . Denote by  $j_{l+2}, \dots, j_{l+1+p} \in \{k+1, \dots, \tilde{k}\}$  the indices determined by  $(\tilde{\lambda}^2)_{ij_0}$ , where  $i = m+t+1, \dots, n$ . By the definition of the extension and Remark 2.2, we obtain for  $h = l+2, \dots, l+1+p$

$$(3.11) \quad \left. \frac{\partial \Pi_{\tilde{V}_{\tilde{\lambda}}}(z)}{\partial \tilde{\lambda}_{j_h}} \right|_{\tilde{\lambda}=\tilde{\lambda}^0} = z_{j_0} \hat{e}_i \in V_{\lambda^0}^\perp \cap \tilde{V}_{\tilde{\lambda}^0},$$

where  $i$  is determined by  $h$ . Let  $\xi = u_{j_1} \wedge \dots \wedge u_{j_{l+1+p}}$ , where  $\{u_1, \dots, u_{\tilde{k}}\}$  is the natural basis of  $\mathbb{R}^{\tilde{k}}$ . Now (3.10), (3.11), (2.5) and the fact that  $|z_{j_0}| \geq \frac{|z|}{\sqrt{t}}$  combine to give for  $r = l+1+p$  that

$$\begin{aligned} \|\wedge_r D_{\tilde{\lambda}} \Pi_{\tilde{V}_{\tilde{\lambda}^0}}(z)\| &\geq \|\wedge_r D_{\tilde{\lambda}} \Pi_{\tilde{V}_{\tilde{\lambda}^0}}(z)(\xi)\| \\ &= \|A_{j_1}(z) \wedge \dots \wedge A_{j_{l+1}}(z)\| \cdot \|z_{j_0} \hat{e}_{m+t+1} \wedge \dots \wedge z_{j_0} \hat{e}_n\| > \frac{d'}{(\sqrt{t})^p} |z|^{l+1+p}. \end{aligned}$$

Hence, the assumptions of Proposition 3.6 are valid for the extended family  $\{\Pi_{\tilde{V}_{\tilde{\lambda}}} \mid \tilde{\lambda} \in \tilde{\Lambda}\}$  (the bounds are uniform since we consider only the compact set  $\overline{B}(\lambda^0, R')$ ).

Applying Proposition 3.6 to the family  $\{\Pi_{\tilde{V}_{\tilde{\lambda}}} \mid \tilde{\lambda} \in \tilde{\Lambda}\}$  implies that the assumptions of Lemma 2.1 are valid for the family  $\{\Pi_{\tilde{V}_{\tilde{\lambda}}} \mid \tilde{\lambda} \in B(\tilde{\lambda}^0, R) \cap \tilde{\Lambda}\}$ , where  $R$  is as in Proposition 3.6. Under the assumption  $\dim \mu \leq r$  Lemma 2.1 gives  $\dim(\Pi_{\tilde{V}_{\tilde{\lambda}}})_* \mu = \dim \mu$  for  $\mathcal{L}^{\tilde{k}}$ -almost all  $\tilde{\lambda} \in B(\tilde{\lambda}^0, R) \cap \tilde{\Lambda}$ . Moreover, from (2.4)

we deduce that

$$(3.12) \quad \dim(\Pi_{V_{\tilde{\lambda}^1}} \circ \Pi_{\tilde{V}_{\tilde{\lambda}}})_* \mu \geq \dim(\Pi_{\tilde{V}_{\tilde{\lambda}}})_* \mu - p$$

for every  $\tilde{\lambda}$ . Observing that

$$(3.13) \quad \Pi_{V_{\tilde{\lambda}^1}} = \Pi_{V_{\tilde{\lambda}^1}} \circ \Pi_{\tilde{V}_{\tilde{\lambda}}},$$

the first inequality in (3.2) follows from Fubini's theorem. The second inequality in (3.2) can be verified similarly: By Lemma 2.1 we have  $\dim(\Pi_{\tilde{V}_{\tilde{\lambda}}})_* \mu \geq r$  for  $\mathcal{L}^{\tilde{k}}$ -almost all  $\tilde{\lambda} \in B(\tilde{\lambda}^0, R) \cap \tilde{\Lambda}$  provided that  $\dim \mu > r$ . As before, (3.12), (3.13) and Fubini's theorem combine to give the second inequality in (3.2). Finally, assuming that  $\dim \mu > p(m-1) + m$ , we get from Lemma 2.1 that for  $\mathcal{L}^{\tilde{k}}$ -almost all  $\tilde{\lambda} \in B(\tilde{\lambda}^0, R) \cap \tilde{\Lambda}$  the projected measure  $(\Pi_{\tilde{V}_{\tilde{\lambda}}})_* \mu$  is absolutely continuous with respect to  $\mathcal{H}^{p(m-1)+m}$ , and therefore,  $(\Pi_{V_{\tilde{\lambda}^1}} \circ \Pi_{\tilde{V}_{\tilde{\lambda}}})_* \mu$  is absolutely continuous with respect to  $\mathcal{H}^m$ . Again, the claim follows from (3.12) and Fubini's theorem.

It remains to prove that the lower bounds and the condition for the absolute continuity are the best possible ones. Let  $l$ ,  $p$  and  $k$  be as in (3.9) and fix  $0 \leq s \leq 1$ . We start by constructing a  $k$ -dimensional family  $\{\Pi_{V_{\lambda}} \mid \lambda \in \Lambda\}$  of projections and a measure  $\mu$  on  $\mathbb{R}^n$  with  $\dim \mu = p + l + s$  such that

$$\dim(\Pi_{V_{\lambda}})_* \mu = \dim \mu - p$$

for  $\mathcal{L}^k$ -almost all  $\lambda \in \Lambda$ . Let  $\Lambda = ] - \delta, \delta[{}^k$  and consider a family  $\{\Pi_{V_{\lambda}} \mid \lambda \in \Lambda\}$  constructed similarly as the above extension using the rotations illustrated in Figure 1. Define  $\mu = \nu_1 \times \nu_2$  where  $\nu_1$  is a  $s$ -dimensional measure on the space spanned by  $e_{l+1}$  and  $\nu_2$  is the restriction of  $\mathcal{L}^{l+p}$  to the unit ball of the space  $X = \langle e_1, \dots, e_l, e_{n-p+1}, \dots, e_n \rangle$ . Then  $\dim \mu = l + p + s$ . Since  $e_j(\lambda) \in X^\perp$  for all  $\lambda \in \Lambda$  and for all  $j = l+1, \dots, m$ , we have  $(\Pi_{V_{\lambda}})_* \nu_2 = (\Pi_{W_{\lambda}})_* \nu_2$  where  $W_{\lambda} = \langle e_1(\lambda), \dots, e_l(\lambda) \rangle$ , and therefore,  $\dim(\Pi_{V_{\lambda}})_* \nu_2 \leq l$  for all  $\lambda \in \Lambda$ . The fact that  $\dim(\Pi_{V_{\lambda}})_* \nu_1 \leq s$  gives  $\dim(\Pi_{V_{\lambda}})_* \mu \leq l + s$  for all  $\lambda \in \Lambda$  implying the sharpness of the first inequality in (3.2).

The sharpness of the second inequality in (3.2) is verified similarly by letting  $\mu$  to be any measure on  $X$  with  $p(l-1) + l \leq \dim_{\mathbb{H}} \mu \leq p(l) + l$ . Finally, if  $l = m-1$ , define  $\mu = \nu_1 \times \nu_2$  where  $\nu_1$  is the 1-dimensional Hausdorff measure restricted to the 1-dimensional four corner Cantor set in  $\langle e_m, e_{m+1} \rangle$  and  $\nu_2$  is as above. Then for  $\mathcal{L}^k$ -almost all  $\lambda \in \Lambda$  the projection of  $\mu$  to  $\langle e_m(\lambda) \rangle$  is singular with respect to  $\mathcal{H}^1$  on  $\langle e_m(\lambda) \rangle$ , implying the singularity of  $(\Pi_{V_{\lambda}})_* \mu$  with respect to  $\mathcal{H}^m$  on  $V_{\lambda}$ .  $\square$

*Remark 3.8.* a) The lower bounds given in Theorem 3.2 are the best possible ones in the sense that for each  $d$  there exist a measure  $\mu$  with  $\dim \mu = d$  and a family of projections such that the lower bounds are achieved. However, this does not mean that for any family and any  $d$  one could construct such a measure.

Different families have different lower bounds - even in the case  $k = 1$ , see [JJLL, Remark 3.5].

b) In the setting of [Fa2] the study of non-existence of Besicovitch  $(n, k)$ -sets leads to a  $k$ -dimensional family of projections from  $\mathbb{R}^{(k+1)(n-k)}$  onto  $\mathbb{R}^{n-k}$ . The set (or the measure) one is projecting is  $k(n - k)$ -dimensional. The essential step is to show that projections have positive measure for almost all parameters. In the case of Besicovitch  $(n, n - 1)$ -sets  $k = n - 1$ , which leads to an  $(n - 1)$ -dimensional family of projections from  $\mathbb{R}^n$  onto  $\mathbb{R}^1$ . The set one is projecting is  $(n - 1)$ -dimensional. According to Theorem 3.2, projections have positive measure provided that  $n - 1 > 1$  implying that there are no Besicovitch  $(n, n - 1)$ -sets for  $n \geq 3$ . For other values of  $k$  the dimension of the parameter space is too small in order to apply Theorem 3.2.

There exist valid proofs for the non-existence of Besicovitch  $(n, k)$ -sets for  $k > \frac{n}{2}$  by Falconer [Fa1] and for  $2^{k-1} + k \geq n \geq 3$  by Bourgain [B] (see also [O]). However, the method of [Fa2] would be more elementary in the sense that it does not use Fourier transform.

#### 4. FAMILIES OF SMOOTH MAPS

In this section we discuss the extension of Theorem 3.2 to families of smooth maps from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ . Note that an orthogonal projection is uniquely determined by its kernel, and moreover, the restriction of a linear map to the orthogonal complement of its kernel is a diffeomorphism onto its image. These simple observations lead to the following definition.

**Definition 4.1.** Let  $\Lambda \subset \mathbb{R}^k$  be open and let  $\mathcal{F} = \{F_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m \mid \lambda \in \Lambda\}$  be a family of  $C^2$ -maps. Define  $V_\lambda^x := \ker(D_x F_\lambda(x))^\perp$ . The family  $\mathcal{F}$  is *non-degenerate* if the following conditions are satisfied:

- (1) The plane  $V_\lambda^x$  is  $m$ -dimensional for all  $x \in \mathbb{R}^n$  and  $\lambda \in \Lambda$  and the family  $\{\Pi_{V_\lambda^x} \mid \lambda \in \Lambda\}$  is non-degenerate for all  $x \in \mathbb{R}^n$ .
- (2) The map  $x \mapsto D_\lambda \Pi_{V_\lambda^x}$  is continuous.
- (3) There exist constants  $C_1, C_2 > 0$  such that  $\|D_\lambda D_x F_\lambda(x)\| \leq C_1$  and  $\|D_\lambda D_x^2 F_\lambda(x)\| \leq C_2$  for all  $x \in \mathbb{R}^n$  and  $\lambda \in \Lambda$ .

Here the derivatives with respect to  $x$  and  $\lambda$  are denoted by  $D_x$  and  $D_\lambda$ , respectively, and the norm of a linear map is denoted by  $\|\cdot\|$ .

*Remark 4.2.* Restricting our consideration to compact sets  $K_1 \subset \Lambda$  and  $K_2 \subset \mathbb{R}^n$ , we may assume that the constants in Definition 4.1 are independent of  $x$ , the map  $x \mapsto D_\lambda \Pi_{V_\lambda^x}$  is uniformly continuous and there exists a constant  $d > 0$  such that  $|\det D_x F_\lambda(x)|_{V_\lambda^x}| > d$ . Condition (3) is valid if  $D_\lambda D_x F(\cdot)$  and  $D_\lambda D_x^2 F(\cdot)$  are assumed to be continuous.

**Theorem 4.3.** Let  $\Lambda \subset \mathbb{R}^k$  be an open set and let  $\mu$  be a finite Radon measure on  $\mathbb{R}^n$  with compact support. Assume that the family  $\mathcal{F} = \{F_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m \mid \lambda \in \Lambda\}$

of  $C^2$ -maps is non-degenerate. Then for all  $l = 0, \dots, m-1$  and for  $\mathcal{L}^k$ -almost all  $\lambda \in \Lambda$

$$(4.1) \quad \dim(F_\lambda)_*\mu \geq \begin{cases} \dim \mu - p(l), & \text{if } p(l) + l \leq \dim \mu \leq p(l) + l + 1, \\ l + 1, & \text{if } p(l) + l + 1 \leq \dim \mu \leq p(l + 1) + l + 1, \end{cases}$$

where  $p(l)$  is as in (3.1). Furthermore, for  $\mathcal{L}^k$ -almost all  $\lambda \in \Lambda$  the image measure  $(F_\lambda)_*\mu$  is absolutely continuous with respect to  $\mathcal{H}^m$  provided that  $\dim \mu > p(m-1) + m$ . The lower bounds given in (4.1) and the condition for the absolute continuity are the best possible ones.

*Proof.* We proceed as in the proof of Theorem 3.2. The essential step is to define an extended family  $\tilde{F}_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^{m+p}$  for which the assumptions of Lemma 2.1 are valid.

Let  $x^0 \in \mathbb{R}^n$  and  $\lambda^0 \in \Lambda$ . Let  $R' > 0$  and  $\varepsilon > 0$  be sufficiently small. We identify the range of  $F_\lambda$  with  $V_\lambda^{x^0} = \ker(D_x F_\lambda(x^0))^\perp$  such that  $F_\lambda(x^0) = 0 \in V_\lambda^{x^0}$ . As in the proof of Theorem 3.2, the  $k$ -dimensional family of  $m$ -planes  $\{V_\lambda^{x^0} \mid \lambda \in B(\lambda^0, R')\}$  is extended to a  $\tilde{k}$ -dimensional family of  $(m+p)$ -planes  $\{\tilde{V}_\lambda \mid \tilde{\lambda} \in \tilde{\Lambda}\}$ . Denoting the orthogonal complement of  $V_\lambda^{x^0}$  inside  $\tilde{V}_\lambda$  by  $V_\lambda^N$  and identifying  $\mathbb{R}^{m+p}$ ,  $V_\lambda^{x^0} \times V_\lambda^N$  and  $\tilde{V}_\lambda$  with each other, the extended family  $\tilde{\mathcal{F}} = \{\tilde{F}_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^{m+p} \mid \tilde{\lambda} \in \tilde{\Lambda}\}$  is defined by  $\tilde{F}_\lambda(y) = (F_\lambda(y), \Pi_{V_\lambda^N}(y))$ .

It is enough to prove that the assumption (2.6) is satisfied for all  $x, y \in B(x^0, \varepsilon)$  such that  $|\tilde{F}_\lambda(y) - \tilde{F}_\lambda(x)| \leq \tilde{\delta}|y - x|$  for some small  $\tilde{\delta}$ . Writing  $G(\tilde{\lambda}) = \tilde{F}_\lambda(y) - \tilde{F}_\lambda(x)$  and observing that  $D_x F_\lambda(x)(v) = D_x F_\lambda(x)(\Pi_{V_\lambda^x}(v))$  for any  $v \in \mathbb{R}^n$ , we have by Taylor's formula

$$\begin{aligned} D_{\tilde{\lambda}} G(\tilde{\lambda}) &= \left( D_{\tilde{\lambda}}(D_x F_\lambda(x)(\Pi_{V_\lambda^x}(y-x)) + \frac{1}{2} D_x^2 F_\lambda(\xi)(y-x)), D_{\tilde{\lambda}} \Pi_{V_\lambda^N}(y-x) \right) \\ &= \left( D_x F_\lambda(x)(D_{\tilde{\lambda}} \Pi_{V_\lambda^{x^0}}(y-x) + D_{\tilde{\lambda}} \Pi_{V_\lambda^x}(y-x) - D_{\tilde{\lambda}} \Pi_{V_\lambda^{x^0}}(y-x)) \right. \\ &\quad \left. + D_{\tilde{\lambda}} D_x F_\lambda(x)(\Pi_{V_\lambda^x}(y-x)) + \frac{1}{2} D_{\tilde{\lambda}} D_x^2 F_\lambda(\xi)(y-x), D_{\tilde{\lambda}} \Pi_{V_\lambda^N}(y-x) \right) \\ &= (D_x F_\lambda(x) \oplus \text{Id})(D_{\tilde{\lambda}} \Pi_{\tilde{V}_\lambda}(y-x)) \\ &\quad + \left( D_x F_\lambda(x)(D_{\tilde{\lambda}} \Pi_{V_\lambda^x}(y-x) - D_{\tilde{\lambda}} \Pi_{V_\lambda^{x^0}}(y-x)) \right. \\ &\quad \left. + D_{\tilde{\lambda}} D_x F_\lambda(x)(\Pi_{V_\lambda^x}(y-x)) + \frac{1}{2} D_{\tilde{\lambda}} D_x^2 F_\lambda(\xi)(y-x), 0 \right). \end{aligned}$$

Note that in the above sum the norm of the second term does not change when replacing  $D_{\tilde{\lambda}}$  by  $D_\lambda$ . By Remark 4.2, we have  $|\Pi_{V_\lambda^x}(y-x)| \leq \tilde{d}\tilde{\delta}|y-x|$  for some  $\tilde{d}$ , and therefore, by Definition 4.1, the norm of the second term is less than  $\tilde{\varepsilon}|y-x|$  where  $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon, \tilde{\delta})$  tends to zero as  $\varepsilon$  and  $\tilde{\delta}$  tend to zero. This, in turn, implies that  $D_{\tilde{\lambda}} G(\tilde{\lambda})$  is a small perturbation of a diffeomorphic image of

$D_{\tilde{\lambda}}\Pi_{\tilde{V}_{\tilde{\lambda}}}(y-x)$ . According to the proof of Proposition 3.6, the singular values of  $D_{\tilde{\lambda}}\Pi_{\tilde{V}_{\tilde{\lambda}}}(y-x)$  are bounded from above and below when  $\tilde{\lambda}$  is restricted to a suitable subspace. (The restriction is denoted by  $T^{\lambda^1}$  in the proof of Proposition 3.6.) Thus the same is true for  $D_{\tilde{\lambda}}G(\tilde{\lambda})$ , and from [JJLL, Lemma 3.1] we conclude that a suitable restriction of  $G$  is a diffeomorphism with uniform lower and upper bounds. Proceeding as in the proof of Proposition 3.6, we have for all  $\delta > 0$  and for all  $x \neq y \in B(x^0, \varepsilon)$

$$\mathcal{L}^k(\{\lambda \in B(\lambda^0, R) \mid |\tilde{F}_{\tilde{\lambda}}(y) - \tilde{F}_{\tilde{\lambda}}(x)| \leq \delta\}) \leq C\delta^r |y - x|^{-r}.$$

The rest of the proof follows similarly as that of Theorem 3.2.  $\square$

*Remark 4.4.* It is natural to consider whether the part of [PS, Theorem 7.3] concerning the exceptional sets of projections is useful in our setting. For this purpose, one needs to extend a  $k$ -dimensional family  $\{F_{\lambda} : \mathbb{R}^n \rightarrow \mathbb{R}^m \mid \lambda \in \Lambda\}$  to a transversal family  $\{\tilde{F}_{\tilde{\lambda}} : \mathbb{R}^n \rightarrow \mathbb{R}^m \mid \tilde{\lambda} \in \tilde{\Lambda}\}$  for which [PS, Theorem 7.3] may be applied. Usually the extended parameter space  $\tilde{\Lambda}$  is  $m(n-m)$ -dimensional. For the extended family [PS, (7.4)] reads in our notation as follows

$$(4.2) \quad \dim\{\tilde{\lambda} \in \tilde{\Lambda} \mid \dim(\tilde{F}_{\tilde{\lambda}})_*\mu \leq \sigma\} \leq m(n-m) + \sigma - \alpha$$

where  $I_{\alpha}(\mu) < \infty$ . Inequality (4.2) gives a lower bound for  $\mathcal{L}^k$ -almost all  $\lambda \in \Lambda$  provided that  $m(n-m) + \sigma - \alpha < k$ . Recalling (2.3), the best possible lower bound obtained in this way is

$$(4.3) \quad \dim(F_{\lambda})_*\mu \geq \dim \mu - (m(n-m) - k).$$

The lower bound given by Theorem 4.3 is better than (4.3) except in the case where  $k \geq (m-1)(n-m)$  and  $\dim \mu \geq p(m-1) + m - 1$ . In this case (4.3) equals the bound given by Theorem 4.3 but we assume less regularity from the family than [PS, Theorem 7.3]. Similarly, our result gives a better bound than [PS, (7.6)] unless  $k \geq l + m(n-m) - m$ , which implies  $p(l) = 0$ .

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